7.1 Integration by Parts *(continued)*

Summary: $\int u\,dv = uv - \int v\,du$

1. Pick u = ??. The rest is dv.
2. Compute du and v.

Entry Task: Evaluate

1. $\displaystyle\int \frac{\ln(x)}{\sqrt{x}}\,dx$

$u = \ln(x) \quad dv = x^{-1/2}\,dx$

$du = \frac{1}{x}\,dx \quad v = 2x^{1/2} = 2\sqrt{x}$

$= 2\sqrt{x}\ln(x) - \int 2\underbrace{\sqrt{x}\,\frac{1}{x}}_{x^{-1/2}}\,dx$

$= 2\sqrt{x}\ln(x) - 2\left(2\cdot x^{1/2}\right) + C$

$= \boxed{\underbrace{2\sqrt{x}}_{F}\,\underbrace{\ln(x)}_{S} - 4\sqrt{x} + C}$

$\boxed{\text{CHECK}}$

$\overset{F'}{\frac{1}{\sqrt{x}}}\,\overset{S}{\ln(x)} + \overset{F}{2\sqrt{x}}\,\overset{S'}{\frac{1}{x}} - \frac{2}{\sqrt{x}}$ ✓

2. $\displaystyle\int_0^1 x^2 e^{x/3}\,dx$

$u = x^2 \quad dv = e^{\frac{1}{3}x}\,dx$

$du = 2x\,dx \quad v = 3e^{1/3 x}$

$= 3x^2 e^{\frac{1}{3}x}\Big|_0^1 - \int_0^1 6x e^{1/3 x}\,dx$

$= 3e^{1/3} - \left[18xe^{1/3 x}\Big|_0^1 - \int_0^1 18 e^{1/3 x}\,dx\right]$

$u = 6x \quad dv = e^{1/3 x}\,dx$

$du = 6\,dx \quad v = 3e^{1/3 x}$

$= 3e^{1/3} - 18e^{1/3} + 18\cdot 3\, e^{1/3 x}\Big|_0^1$

$= -15e^{1/3} + 54\left(e^{1/3} - e^{0}\right)$

$= \boxed{39e^{1/3} - 54}$

Integration by parts is good for:

Products: $\underset{u}{x}e^{x}, \underset{u}{x^2}\cos(3x), \underset{u}{x}\sin(5x)$

Logs: $\underset{u}{\ln(x)}, x^{10}\underset{u}{\ln(x)}, \frac{\overset{u}{\ln(x)}}{x^3}, \ldots$

Inv. Tri: $\underset{u}{\sin^{-1}(x)}, \mathrm{x}\underset{u}{\tan^{-1}(x)}, \ldots$

Products: $\underset{u}{e^{x}}\sin(x), \underset{u}{e^{x}}\cos(x)$

Example:

$$\int \cancel{\sin^{-1}(x)}\ \tan^{-1}(x)\,dx = \int \cancel{\arcsin(x)}\ \arctan(x)\,dx$$

$u = \tan^{-1}(x) \qquad dv = dx$

$du = \frac{1}{1+x^2}dx \qquad v = x$

$$= x\tan^{-1}(x) - \int \frac{x}{1+x^2}dx$$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$$= x\tan^{-1}(x) - \int \frac{\cancel{x}}{u}\frac{1}{2\cancel{x}}du$$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|u| + c$$

$$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C}$$

ASIDE.

$y = \tan^{-1}(x)$

$\Rightarrow \tan(y) = x$

$\Rightarrow \sec^2(y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)}$

$= \frac{1}{1+\tan^2(y)}$

$= \frac{1}{1+x^2}$

Example: (Never ending integration by parts and how to end it):

$$\int e^x \cos(x)\, dx$$

$u = e^x \qquad dv = \cos(x)dx$

$du = e^x dx \qquad v = \sin(x)$

$e^x \sin(x) - \int e^x \sin(x) dx$

$u = e^x \qquad dv = \sin(x)dx$

$du = e^x dx \qquad v = -\cos(x)$

$e^x \sin(x) - \left(-e^x \cos(x) - \int e^x(-\cos(x))dx\right)$

$\Rightarrow \quad \int e^x \cos(x)dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)dx$

$2\int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) + C_0$

$\int e^x \cos(x) dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + \frac{C_0}{2}$

7.2 Trigonometric Integral Methods

Goal: A procedure to integrate *any* combination of trig functions.

Motivating examples: ***These are substitution problems, what is u?***

$$\int \underset{u}{\sin^3(x)}\,(1-\underset{u}{\sin^2(x)})\,\underbrace{\cos(x)\,dx}_{du}$$

$$\int (1-\underset{u}{\cos^2(x)})\,\underset{u}{\cos^5(x)}\,\underbrace{\sin(x)\,dx}_{-du}$$

$$\int \underset{u}{\tan^5(x)}\,(1+\underset{u}{\tan^2(x)})\,\underbrace{\sec^2(x)\,dx}_{du}$$

$$\int \underset{u}{\sec^6(x)}\,\underbrace{\sec(x)\tan(x)\,dx}_{du}$$

7.2 idea: Use trig identities to turn almost all trig problems into one of these situations!

Essential Tools

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \cot(x) = \frac{\cos(x)}{\sin(x)},$$
$$\sec(x) = \frac{1}{\cos(x)}, \csc(x) = \frac{1}{\sin(x)}.$$

See my online postings (or the Appendix of your book) for a more general discussion and proofs of trig identities.

$$\sin^2(x) + \cos^2(x) = 1$$
$$\tan^2(x) + 1 = \sec^2(x)$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$
$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$
$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

Case 1 (cosine or sine has an odd power)

$i) \int \sin^2(x)\cos^3(x)\,dx$ ← ODD

PULL OUT ONE COSINE

$\int \sin^2(x)\cos^2(x)\cos(x)\,dx$

USE IDENTITY

$\int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$

$u=\sin(x)$

$du=\cos(x)\,dx$

$\int u^2(1-u^2)\,du$

$\int u^2-u^4\,du = \frac{1}{3}u^3-\frac{1}{5}u^5+c = \boxed{\frac{1}{3}\sin^3(x)-\frac{1}{5}\sin^5(x)+c}$

$ii) \int \sin^3(x)\,dx$ ← ODD

PULL OUT ONE SINE

$\int \sin^2(x)\sin(x)\,dx$

USE IDENTITY

$\int (1-\cos^2(x))\sin(x)\,dx$

$u=\cos(x)$

$du=-\sin(x)\,dx$

$-\frac{1}{\sin(x)}du=dx$

$\int (1-u^2)\cancel{\sin(x)}\frac{1}{-\cancel{\sin(x)}}du$

$-\int 1-u^2\,du$

$-(u-\frac{1}{3}u^3)+C$

$\boxed{-\cos(x)+\frac{1}{3}\cos^3(x)+c}$

Case 2 (Both have even powers)

$i) \int \cos^2(x)\,dx$ ← even

$\int \frac{1}{2}(1+\cos(2x))\,dx$

$\frac{1}{2}\int 1+\cos(2x)\,dx$

$\frac{1}{2}(\theta+\frac{1}{2}\sin(2x))+C$

$\boxed{\frac{1}{2}\theta+\frac{1}{4}\sin(2x)+C}$

$ii) \int \sin^4(x)\,dx$

$\int \sin^2(x)\sin^2(x)\,dx$

$=\int \frac{1}{2}(1-\cos(2\theta))\frac{1}{2}(1-\cos(2\theta))\,d\theta$

$=\frac{1}{4}\int 1-2\cos(2\theta)+\cos^2(2\theta)\,d\theta$

$=\frac{1}{4}\left[\theta-\sin(2\theta)+\int\frac{1}{2}(1+\cos(4\theta))\,d\theta\right]$

$=\frac{1}{4}\theta-\frac{1}{4}\sin(2\theta)+\frac{1}{8}(\theta+\frac{1}{4}\sin(4\theta))+C$

$=\boxed{\frac{3}{8}\theta-\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta)+C}$

Case 3 (even power on secant)

$$\int \tan^2(x)\,\sec^4(x)\,dx$$

even

PULL OUT $\sec^2(x)$

$$\int \tan^2(x)\sec^2(x)\sec^2(x)\,dx$$

$$\int \tan^2(x)\left(1+\tan^2(x)\right)\sec^2(x)\,dx$$

$u=\tan(x)$

$du=\sec^2(x)\,dx$

$$\int u^2(1+u^2)\,du$$

$$\int u^2+u^4\,du$$

$$\frac{1}{3}u^3+\frac{1}{5}u^5+C$$

$$=\boxed{\frac{1}{3}\tan^3(\theta)+\frac{1}{5}\tan^5(\theta)+C}$$

Case 4 (Odd power on ~~secant~~ tangent, and at least one ~~tangent~~ sec(x))

$$\int \tan^3(x)\,\sec^5(x)\,dx$$

odd

PULL OUT $\sec(x)\tan(x)$

$$\int \tan^2(x)\sec^4(x)\quad \sec(x)\tan(x)\,dx$$

$$\int \left(\sec^2(x)-1\right)\sec^4(x)\sec(x)\tan(x)\,dx$$

$u=\sec(x)$

$du=\sec(x)\tan(x)\,dx$

$$\int (u^2-1)\,u^4\,du$$

$$\int u^6-u^4\,du$$

$$\frac{1}{7}u^7-\frac{1}{5}u^7+C$$

$$\boxed{\frac{1}{7}\sec^7(x)-\frac{1}{5}\sec^7(x)+C}$$

Notes:

And if you've tried all methods and are stuck, here are things to try:

1. Rewrite in terms of sin(x) and cos(x).
2. Rewrite in terms of sec(x) and tan(x).
3. Try using trig identities.

And there are still a few "holes".
Particularly, odd power on sec(x).
For these you can quote
(proof in the book):

$$\boxed{\begin{gathered}\int \tan(x)\,dx = \ln|\sec(x)| + C \\[2em] \int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C \\[2em] \int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(\mathrm{x}) + \tan(\mathrm{x})| + C\end{gathered}}$$

What is the first step in each integral below?

ODD

$$\int \sin^3(x)\cos^4(x)dx = \int (1-\cos^2(x))\cos^4(x)\sin(x)dx$$

ODD

$$\int \sin^5(x)\cos^3(x)dx = \int \sin^5(x)(1-\sin^2(x))\cos(x)dx$$

EVEN

$$\int \cos^4(x)dx = \int \cos^2(x)\cos^2(x)dx = \int \tfrac{1}{2}(1+\cos(2x))\,\tfrac{1}{2}(1+\cos(2x))dx$$

EVEN

$$\int \tan^5(x)\sec^4(x)dx = \int \tan^5(x)(1+\tan^2(x))\sec^2(x)dx$$

ODD

$$\int \tan^5(x)\sec(x)dx = \int (\sec^2(x)-1)^2\sec(x)\tan(x)dx$$